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ON YAKHNO'S MODEL FOR A LEADING CENTER

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ABSTRACT

A reaction-diffusion model suggested by Yakhno as a model for the centers of periodic activity often seen in excitable media such as cardiac tissue and special chemical reagents is subjected to multiple-scale analysis. A fairly complete description of the process (which involves the generation of wave fronts) is obtained, but it is shown to be structurally unstable as it depends on symmetries in the basic equations. The allowed amount of deviation from symmetry is estimated.

AMS (MOS) Subject Classifications: 92A09, 93A40, 35K55, 35B25.

Key Words: Excitable medium, chemical waves, relaxation oscillation, multiple scales, leading center, reaction-diffusion equation, structural stability.

Work Unit 2 - Physical Mathematics

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SIGNIFICANCE AND EXPLANATION

Leading centers are locations often observed in excitable media such as cardiac tissue and special chemical reagents, which apparently serve as the source of wavelike activity such as rotating spiral configurations in the chemical concentration or physico-chemical state. A possible model to explain this is subjected to asymptotic analysis.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON YAKHNO'S MODEL FOR A LE[illegible]ING CENTER

Paul C. Fife

1. Introduction

In [1], Yakhno presented a rather simple class of reaction-diffusion systems which exhibit a phenomenon he called "division" of arrested wave fronts, resulting in the successive generation of diverging pairs of fronts from a central point. These diverging pairs form an antisymmetric composite profile, so differ from such front generation processes studied, for example, in [4], [5], and other places. This mechanism was proposed as a possible model for the center of periodic activity sometimes seen in such excitable media as heart muscle (fibrillation) and the Belousov-Zhabotinskii reagents (especially spiral waves), and sometimes called leading centers. Due to the great interest in such centers, it is appropriate to examine Yakhno's mechanism closely.

The model was further discussed in [2] and [3], where the results of numerical simulation were given. It was brought out in those papers that when the symmetry properties present in Yakhno's initial example are violated, the mechanism he described might not occur, due to "running away" of the wave fron[illegible] [illegible]efore it has a chance to divide.

The purpose of the present paper is to give a qualitative analytical treatment (without computer simulation) of Yakhno's model, more detailed than that in the above references. In particular, scaling and order-of-magnitude arguments, which were not fully exploited there, are used throughout the paper. In our description of the division process, the known global stability properties of fronts and similar structures are an important part of the argument. Finally, the question of whether an arrested front (which serves as initial condition in the processes analyzed in [1-3]) can arise in the medium from initial data of some degree of generality is addressed in sec. 4.

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The general conclusion is that Yakhno's mechanism is rather structurally unstable. This appears to agree with the tenor of the discussion in [2] and [3]. The equations originally proposed and analyzed by Yakhno have some symmetry properties; they also have two natural time scales (a fast and a slow one). If symmetry is now violated in the equations, its violation has to be by a small amount, in some sense comparable to ϵ (the ratio of time units in the two scales), in order for the first "division" to take place. The cumulative effect of these deviations would make subsequent divisions even more unlikely. But even beyond this, our analysis of the effect of different initial data is also discouraging. Because of the relaxational nature of the basic equations, initial data will typically rapidly equilibrate to the shape of a wave front, whose trigger velocity is a function of the value of the slow variable v at the front. If v equals a special value v_0 at the front, we have the arrested front described in the cited references. But for $v \neq v_0$ there, we show in sec. 4 that regardless of the distribution of v away from the front, the latter cannot later evolve into an arrested front, which is necessary for division to occur.

The effect of the size of the spatial domain was considered in [3], but ignored here; for simplicity we take it to be infinite.

Calculations reported in [2] suggest that when the perturbation from symmetry in f is large enough, conditions suitable for division may be restored. No explanation has been advanced for such a phenomenon.

Finally, despite the structural instability of Yakhno's mechanism, it is entirely possible that a more complex model, in which a control process is adjoined to the basic equations, may produce the required stability.

I am indebted to Art Winfree for stimulating my interest in this mechanism.

2. The model in the presence of symmetry.

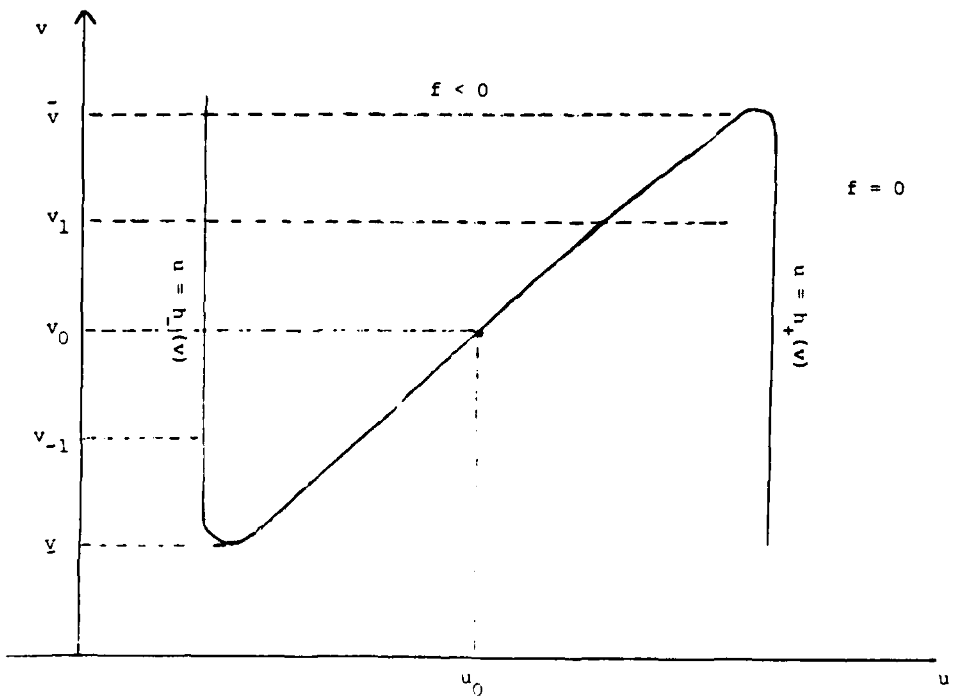
The model Yakhno proposed is a pair of reaction-diffusion equations

$$\varepsilon u_t = \alpha^2 u_{xx} + f(u,v) \tag{1}$$

$$v_t = g(u,v) \quad , \tag{2}$$

where $0 < \varepsilon << 1$, f has the nullcline shown in Figure 1, and g is discontinuous in u at u_0 (this was not assumed in later analysis), changing sign there from $(-)$ to $(+)$. For the qualitative results in this section, there is no loss of generality in taking

$$g(u) = \begin{cases} 1 & \text{for } u > u_0 \\ -1 & \text{for } u < u_0 \end{cases} \quad . \tag{3}$$



We suppose the spatial domain to be the entire x-axis, $-\infty < x < \infty$, so the spatial variable may be scaled at will; this only affects the value of the coefficient α^2 and the form of the initial data (if it is an initial value problem). The scaling which most readily provides a clear description of the process is that for which $\alpha = \varepsilon$, and we shall accordingly take it to be so.

Besides the symmetry in (3), we also assume in this section that the function f is antisymmetric with respect to the point (u_0, v_0):

$$f(u_0 - \delta u,\ v_0 - \delta v) = -f(u_0 + \delta u,\ v_0 + \delta v) \quad . \tag{4}$$

Finally, we suppose that the initial data also have symmetry properties:

$$u(x,0) = \varphi(x/\varepsilon) \tag{5}$$

$$v(x,0) = v_0 \quad , \tag{6}$$

where $\varphi(\xi)$ is bounded with bounded derivatives,

$$\frac{1}{2}\,[\varphi(\xi) + \varphi(-\xi)] = u_0 \tag{7}$$

and $\xi(\varphi(\xi) - u_0) > 0$ for $\xi \neq 0$.

We shall describe the various stages in the evolution of the solution $u(x,t)$, $v(x,t)$.

Stage 1: Equilibration of the initial data to a stationary wave front (duration $O(\varepsilon)$). Changing to variables $\tau = t/\varepsilon$, $\xi = x/\varepsilon$, we may write (1) (with $\alpha = \varepsilon$) to lowest order as

$$u_\tau = u_{\xi\xi} + f(u, v_0) \quad . \tag{8}$$

Here we have used the fact that v_t is bounded, so in the τ time scale, v does not deviate appreciably from its original value v_0. Since (by symmetry)

$$\int_{h_-(v_0)}^{h_+(v_0)} f(u,v_0)du = 0 \quad ,$$

we know that (8) has a globally stable stationary wave front solution $u = U(\xi)$ satisfying $U(\pm\infty) = h_\pm(v_0)$, $U(0) = u_0$. Furthermore, by [6], the solution of (8) with $u(\xi,0) = \varphi(\xi)$ approaches U uniformly in ξ, as $\tau \to \infty$. So in a τ-time interval of $O(1)$ (t-interval of $O(\varepsilon)$), $u(\xi,\tau)$ will be attracted to a neighborhood of $U(\xi)$.

<u>Stage 2</u>: The evolution of v and the concomitant changes in the profile of u(duration $O(1)$). The wave profile to which u has evolved is such that $u > u_0$ exactly when $x > 0$, and this remains so, because of symmetry. Therefore according to (2), (3), we have

$$v(x,t) = \begin{cases} v_0 + t \quad , & x > 0 \\ v_0 - t \quad , & x < 0 \quad . \end{cases} \tag{9}$$

For each fixed time t during this stage, v is a step function in x.

Again because of the symmetries, it will be clear that $v - v_0$ and $u - u_0$ will continue to be odd functions of x for all later time. For this reason, it suffices to consider the problem only on the half-line $-\infty < x < 0$, with the boundary condition $u(0,t) = u_0$. Of course no such boundary condition can be applied to v, because v is discontinuous at $x = 0$; and no boundary condition is needed in any event, because the differential equation for v involves no x-derivatives. The following result is relevant to the study of this problem. The stability spoken of in the lemma is C^0 - stability on the half-line $(-\infty,0)$.

<u>Lemma</u>. Let $f(u)$ be defined and continuously differentiable on the interval $[0,1]$, with $f(0) = 0$ and $f'(0) < 0$. Let $\beta \in (0,1)$. The problem

$$u_{xx} + f(u) = 0 \quad , \quad -\infty < x < 0 \quad ,$$
$$u(0) = \beta,\ u(-\infty) = 0 \quad , \quad u(x) \in [0,1] \quad ,$$

has a solution if and only if $F(u) \equiv \int_0^u f(s)ds < 0$ for all $u \in (0,\beta)$. At most two solutions exist. If any exist, exactly one is monotone increasing, and it is stable if and only if $F(\beta) < 0$. A second solution, nonmonotone and unstable, exists if and only if $F(u) < 0$ for $u \in (0,\beta]$ and $F(u) = 0$ for some $u \in (\beta,1)$.

If $f(1) = 0$ and $f'(1) < 0$, the analogous statement holds for the boundary value problem $u(-\infty) = 1$, $u(0) = \beta$.

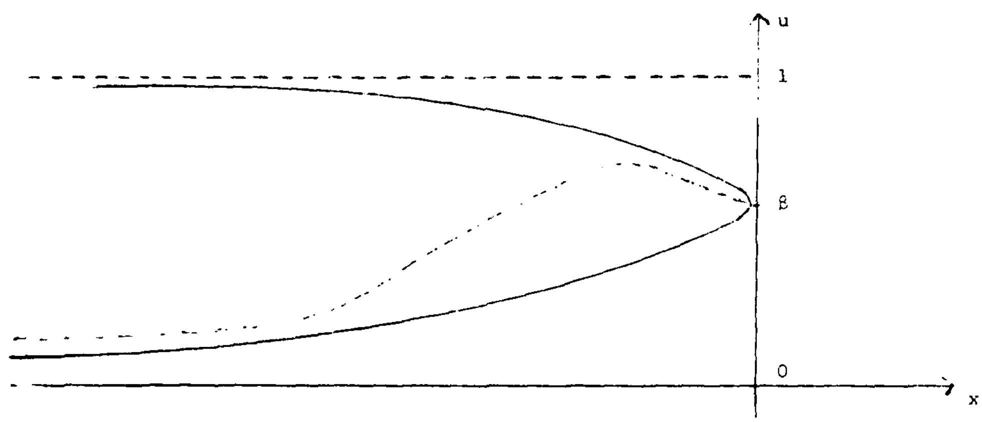


Figure 2. Two stable (solid lines) and one unstable solution in the case $F(u) < 0$, $u \in (0,\beta]$, $F(\beta') = 0$, for some $\beta' \in (\beta,1)$.

The existence part of the lemma follows from a standard phase-plane argument, as in [1]. The instability parts are established by an extension of the reasoning in [7]. The stability is proved in [13].

We apply the lemma to the problem of finding stable stationary solutions $u = U(\xi;v)$, $\xi = \frac{x}{\varepsilon}$ of (1) with $\alpha = \varepsilon$, $v = \text{const}$, on the half-line $(-\infty,0)$, satisfying $U(-\infty;v) = h_-(v)$, $U(0;v) = u_0$.

Referring to Figure 1, we see that there exists such a solution if and only if $v > v_{-1}$, the latter defined by the condition

$$\int_{h_-(v_{-1})}^{u_0} f(u,v_{-1})\,du = 0 \quad .$$

In stage 1, as we have seen, the u-profile is rapidly attracted to $U(\xi;v_0)$. During the present stage, v is steadily evolving according to (9). At the same time, $u(x,t)$ continuously adjusts to the stable profile $U(\xi;\ v_0 - t)$, the characteristic time variable associated with the adjusting process being the fast time τ. In short, during stage 2, for $x < 0$,

$$\begin{aligned} v(x,t) &\sim v_0 - t \quad , \\ u(x,t) &\sim U(\tfrac{x}{\varepsilon};\ v_0 - t) \quad . \end{aligned} \tag{10}$$

This stage continues only so long as $v_0 - t > v_{-1}$. For time $t > v_0 - v_{-1}$, according to the lemma, the equation (1) no longer has a stationary solution satisfying $U(-\infty) = h_-(v)$ (the two lower curves in Figure 2 annihilate themselves).

<u>Stage 3</u>: The division of the front (duration a time interval $<< 1$). For $v < v_{-1}$, the only stationary attractor for (1) satisfying the boundary condition at $x = 0$ is the monotone <u>decreasing</u> solution $u = \hat{U}(\xi;v)$ satisfying $\hat{U}(\xi;v) = h_+(v)$ (the upper curve in Figure 2), and the u-profile is now attracted to that solution. This abrupt flip from one stable state to another is reminiscent of a variety of such events known to occur in chemical engineering and chemical physics, when multiple stable steady states occur ([5, 8, 11], for example). When diffusion is present, the "flip" characteristically generates wave fronts. In the present case, consider a profile for which $U(-\infty,v) = h_-(v)$ which is in the domain of attraction of $\hat{U}(\xi,v)$. For large negative ξ, u takes on values near $h_-(v)$, which is a stable zero for the function $f(u,v)$. The function u

cannot rapidly depart from that neighborhood of $h_-(v)$. However, there is a mechanism for the eventual accomplishment of that goal. If v were held constant (rather than being discontinuous at $x = 0$), there would exist a wave front solution $u(\xi,\tau) = W(\xi - c(v)\tau)$, $c < 0$, satisfying

$$u_\tau = u_{\xi\xi} + f(u,v), \quad u(\pm\infty,\tau) = h_\pm(v)$$

(the negativity of c follows from the fact that the state h_+ is dominant, i.e. $\int_{h_-(v)}^{h_+(v)} f(u,v)du > 0$ in the terminology of [9] (see also [10])). This wave front is very stable [6]. What we will see in Stage 3 is the initiation of a structure which is not precisely this wave front, but (in the $x - t$ coordinate system again) is a sharp front, propagating to the left, attached by a boundary layer to the fixed value u_0 at $x = 0$, as shown in Figure 3. It will also be seen below that the velocity of propagation will not be constant, as v is not constant.

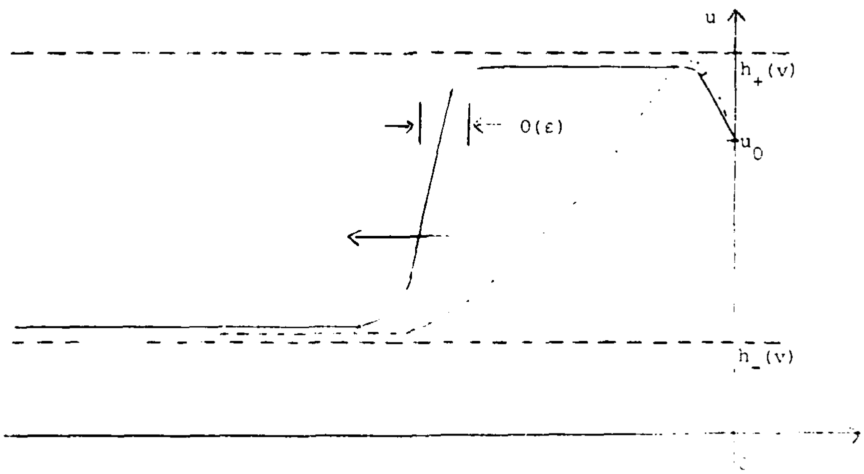


Figure 3

It is intuitively clear (and no doubt provable) that this is a stable configuration, and it is the mode to which the solution "switches" when v become less than v_{-1}. Convergence of solutions to similar structures, with v constant, was studied in [13]. Stage 3 is the initial stage, in which the left-facing front is first formed (the dotted line in Figure 3).

Stage 4: The propagation of the divided front (time scale $O(1)$). The velocity of the propagating wave shown in Figure 3 is seen, by dimensional analysis, to be $O(1)$ in the $(x - t)$ system, as well as in the $(\xi - \tau)$ system. In fact, it is clearly $O(1)$ in the latter system, and so the traveling wave would behave like a function of $\xi - c\tau = \frac{x}{\varepsilon} - c\,\frac{t}{\varepsilon} = \frac{x - ct}{\varepsilon}$ $(c = O(1))$. The width of the front is $O(\varepsilon)$ since the characteristic spatial variable is $\xi = \frac{x}{\varepsilon}$.

We have now (Figure 3) the picture of a sharp front moving to the left with velocity $O(1)$, and at the same time v is evolving at a rate $O(1)$. In fact by (3), v is increasing at a constant rate on the interval between the front and the origin, and decreasing on the left of the front. Letting $y(t)$ be the position of the front (where $u = u_0$, say) and $t_1 = v_0 - v_{-1}$ the time at which stage 3 occurs, we can calculate the approximate position of the front, and the distribution of v, as functions of t as follows. To the left of the front, v is steadily decreasing, so

$$v(x,t) \sim v_{-1} - (t-t_1), \quad x \leq y(t) \quad . \tag{11}$$

The front's velocity c_F is determined by this value of v, since the velocity of the front solution of (8) can depend only on v in that equation [10], which is the (relatively well defined) value of v at the front. As in [4], we therefore have

$$c_F(t) = c(v_{-1} - (t-t_1)) \quad .$$

Note that $c < 0$ and the front accelerates, since $|c|$ is a decreasing function of v (see Appendix in [5]). Integrating, we have

$$y = Y(t) \equiv \int_{t_1}^{t} c_F(s)ds \quad .$$

Inverting this function yields $T(x)$, the time of arrival of the front at position x. Finally, the function v is given by

$$v(x,t) = v_{-1} - (T(x) - t_1) + (t-T(x)), \; y(t) \leq x \leq 0 \quad . \tag{12}$$

Stage 5: The vanishing of the moving front (duration $<< 1$). Eventually, the value of v at the front will reach $\underline{v}$, the minimal value on the nullcurve in Figure 1. For $v < \underline{v}$, $h_-(v)$ no longer exists, and the front can no longer exist. When this happens, u will approach the value $h_+(v)$, uniformly for all $x < \delta < 0$, effectively annihilating the propagating front. The characteristic time scale for this process is $<< 1$. At this point, the distribution of v is given by (12) with $t = t_2$, the time of Stage 5. Of course $v(x,t_2) \equiv \underline{v}$ for $x < y(t_2)$. Notice that $v(x,t_2)$ attains a maximum of $2v_{-1} - \underline{v}$ at $x = 0$.

Stage 6. The reversed stationary front (duration $0(1)$). Now $u > u_0$ everywhere for $x < 0$, and so v increases with uniform velocity 1. Just as in Stage 2, the u-profile continuously adjusts to the increasing value of v. In the boundary layer just to the left of the origin, the sharp variation of u is, in the stretched coordinates, a function decreasing from $h_+(v)$ to u_0. Hence the value of v there is $v(x-0,t) = 2v_{-1} - \underline{v} + (t-t_2) \equiv \bar{v}(t)$. Outside the boundary layer, of course, $f(u,v) \sim 0$, so $u(x,t) \sim h(v(x,t))$, and v is determined as indicated above.

This stage continues until $\bar{v}(t)$ reaches the value v_1.

Stage 7: Second front division (duration $<< 1$). This division produces a downjump front moving to the left, analogous to the upjump produced in Stage 3.

Remark: We have assumed that Stage 5 occurs before Stage 7, but in some cases the order will be reversed. In such cases, there will simultaneously exist an upjump front and a downjump moving to the left, before the front is eliminated. If the function g in (2) is altered so that it depends on u as well, and is such that $g = 0$ for some value

$v^* \in (\underline{v}, v_{-1})$ and $u < u_0$, then Stage 5 will never be reached, and an infinite train of fronts will accumulate.

Of course to the right of the origin, a succession of fronts (of the opposite shape) are generated in the antisymmetric fashion. Thus at all times, the picture is such that a stationary front, facing either left or right, exists at the origin, and in addition fronts may exist which diverge from the origin in opposite directions. The whole process is shown schematically in Figure 4. The profiles shown there are the u-profiles at the end of the various successive stages. The numbers indicate the stages.

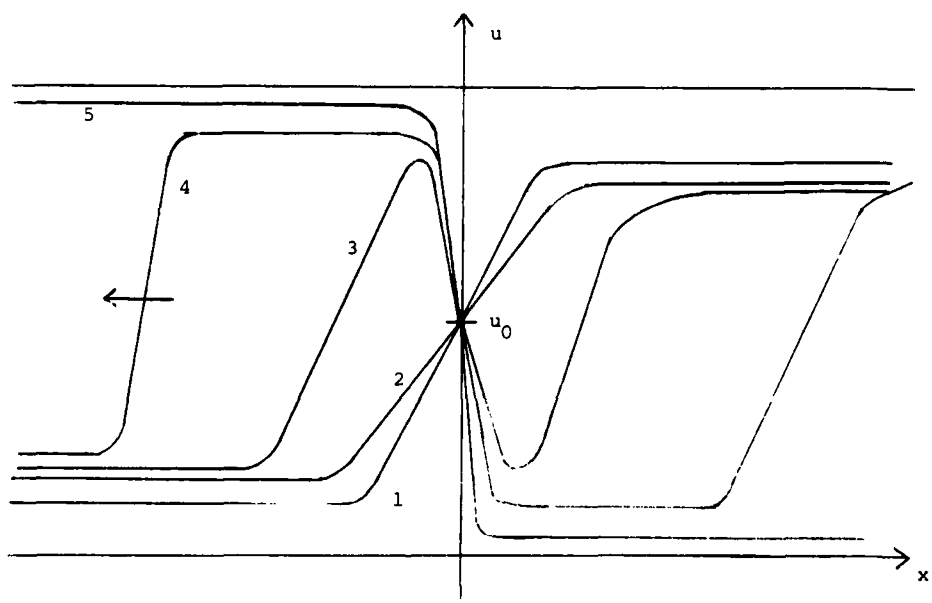


Figure 4

3. The effects of loss of symmetry in f and g

We now consider whether the phenomenon of front division, as described in sec. 2, can still occur when the various special assumptions made there are modified. For this purpose, we first ask what features of the process are the essential ones for front division, and then whether these features are preserved or lost when the symmetry conditions are relaxed.

The most essential feature seems to be that the function $v(x,t)$ be capable of achieving a spatial variation of the order of magnitude 1 within the confines of a single wave front. In the symmetric case, v has a jump discontinuity of this magnitude exactly at the center of the front, which has width $O(\varepsilon)$, and so that property holds. Of course, it is not necessary for v to be discontinuous, but it should at least vary by such an amount in an interval of width $O(\varepsilon)$ centered at the front. In fact, if it does not, then to first approximation, v may be considered constant through the front, and the front would represent a horizontal transition in Figure 1 from the branch h_- to h_+. Such fronts always exist, whereas it is precisely the nonexistence of an appropriate front that results in the phenomenon of division.

We shall be examining various ways to relax our previous assumptions, and shall ask whether the above necessary condition continues to hold. Throughout, we retain (1) with $\alpha = \varepsilon$ as the fundamental equation. The presence of the factor $\varepsilon << 1$ with u_t makes u a fast variable. The role of this parameter in the foregoing analysis was to make the u-profile respond quickly to changes in the function f. Thus, when v reaches the critical value v_{-1}, this quick response initiates the front division before v has a chance to attain the other critical value $\underline{v}$, after which time a front reversal would occur, but no division. Of course other mechanisms may delay v's reaching that second point. Calculations reported in [3] indicate division may indeed occur even if $\varepsilon \sim 1$. But the smallness of ε enhances the division process, and we shall retain that assumption throughout the following.

The parameter ε^2 as diffusion coefficient in (1) can always be obtained by proper x-scaling, and we retain that also. The only real effect of such scaling may be in regard to the initial data and Stage 1; more comments on this will be given in the next section.

(I) Relaxation of the symmetry in (3). In place of (3), assume (again for simplicity)

$$g(u) = \begin{cases} 1 + s & , \; u > u_0 \\ -1 & , \; u < u_0 \end{cases} \quad , \tag{13}$$

for some constant $s > 0$. The ideas here are immediately extendable to the case of more general discontinuous function $g(u,v)$. This change does not affect Stage 1. The effect here is to cause the front, previously fixed during Stage 2 at $x = 0$, to move. In fact if it were not to move for $t < t_0$, then at that time the last term in (1) could be expressed as

$$\hat{f}(u) \equiv \begin{cases} f(u, v_0 - t_0) & , \; u < u_0 \\ f(u, v_0 + t_0(1+s)) & , \; u \quad u_0 \end{cases} \quad ,$$

This function satisfies $\int_{h_-(v_0 - t_0)}^{h_+(v_0 + t_0(1+s))} f(u)du < 0$, hence h_- is the dominant state, and the front will be moving to the right. So it cannot be that the front was stationary; the effect of the term s is to cause it to travel to the right.

Let $x = y(t) > 0$ be the position of the front (where $u = u_0$). A further consequence of its movement is that v is no longer discontinuous. In fact since the motion is to the right, the value of v at the front's position during Stage 2 is

$$v(y(t),t) = v_0 + t(1+s) \quad ,$$

whereas $v(0,t) = v_0 - t$ for the same reason as before. Between the two values 0 and $y(t)$, v is a continuous monotone function which could be calculated. This continuous function is larger than the discontinuous function equal to $v_0 - t$ for $x < y(t)$ and

$v_0 + t(1+s)$ for $x > v't$. Now f is a decreasing function of v, and the front velocity is a decreasing function of f (see [5], Appendix), so the effect of the motion is to further increase the velocity, adding on [illegible]. This acceleration shows that the asymmetry has a destabilizing effect on the front. The front's velocity during Stage 2 will have magnitude $O(s)$ $(s \to 0)$.

We must ask how far the front will move during Stage 2. Since this stage is of duration $O(1)$, the distance will be $O(s)$. As seen from the above argument, this interval in x between the first and final positions of the front in Stage 2 is where all of the spatial variation in v is concentrated. So the gradient $v_x \approx O(\frac{1}{s})$. To satisfy the condition that the variation be $O(1)$ in a front's width, however, we must have $|v_x| \geq O(\frac{1}{\epsilon})$. This is only possible if $s \leq O(\epsilon)$.

We conclude that under (13), front division is only possible if $s \leq O(\epsilon)$. If this condition is fulfilled, division may occur the first time, but of course subsequent divisions will become more unlikely.

(II) Relaxation of the symmetry (4) in f. In this case, even though (3) is retained, it will no longer necessarily be true that the front's velocity is zero during Stage 2. For example, consider a nullcline as follows:

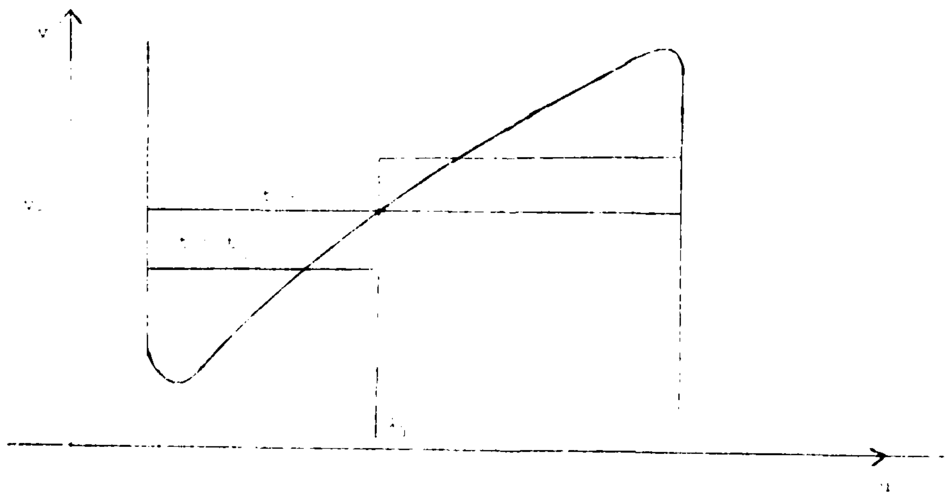


Figure 5

If the front were artificially held fixed for $0 < t < t_0$, the asymmetry here would try to push the front forward, because $\int f\,du$ over the discontinuous line shown for $t = t_0$ will be negative. This forces the front to move. As before, it can be seen that the front's movement further causes acceleration, so the asymmetry is a destabilizing influence. Let s be some measure of asymmetry in f; for example,

$$s = \operatorname{Max}\left| \int_{h_-(v_0 - \omega)}^{u_0} f(u, v_0 - \omega)\,du + \int_{u_0}^{h_+(v_0 + \omega)} f(u, v_0 + \omega)\,du \right| ,$$

the maximum being over all ω for which $h_\pm(v_0 \pm \omega)$ are both defined.

Then the argument in Case (I) can be repeated to show that front division is only possible if $s \leq O(\varepsilon)$.

4. <u>Effect of loss of symmetry in the initial data, and the effect of scaling.</u>

In place of (5) and (6), take

$$u(x,0) = \varphi(x/\varepsilon) \quad ,$$
$$v(x,0) = \psi(x/\varepsilon) \quad . \tag{14}$$

Consider the "initial" curve Γ in the $u - v$ plane (Figure 6) forming the image of these initial conditions: $u = \varphi(\xi)$, $v = \psi(\xi)$:

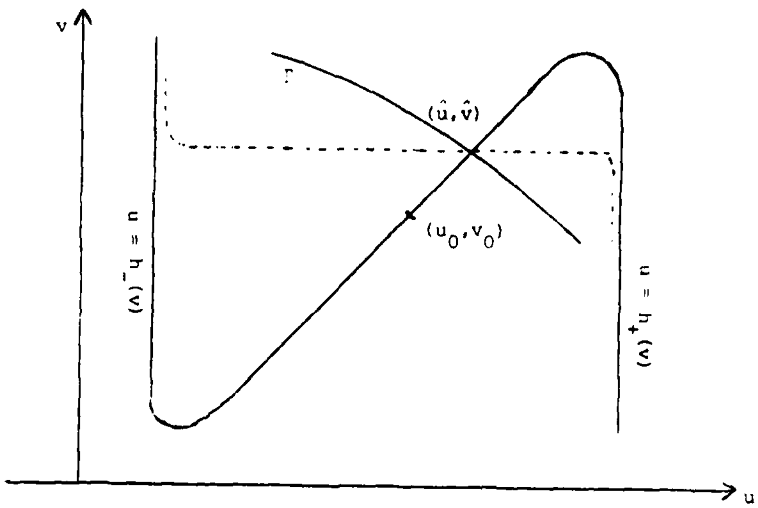


Figure 6

During Stage 1, in time $O(\varepsilon)$ the curve Γ, except in a neighborhood of where it crosses the ascending branch in Figure 6, is drawn to the lines $h_{\pm}$. This forms the initial front profile. If Γ does not cross that branch, there will be no front formed during Stage 1, and so we assume it crosses.

The fact that v is not constant initially as in (6) will have little effect at this stage. What is more important is the place $(\hat{u},\hat{v})$ where Γ crosses the nullcurve. It was assumed previously that $(\hat{u},\hat{v}) = (u_0,v_0)$: now we assume this is not so. For the sake of definiteness, suppose $\hat{v} > v_0$ and that ψ is a decreasing function of x (as indicated in Figure 6). Since h_- dominates at $v = \hat{v}$, we have $c(\hat{v}) > 0$, so at the outset in Stage 2, the front moves to the right at a nonzero speed.

Let us see what happens in the simple case that $\psi(x)$ is linear, so $\psi'(x) = -m < 0$. Let $y(t)$, as before, be the position of the front. Ahead of it, $v(x,t) = \psi(x) + t$, $x > y(t)$, hence $v^+(t) \equiv v(y(t),t)$ satisfies

$$\dot{v}^+ = \psi'(y)\,\dot{y} + 1 = -m\,c + 1 \quad . \tag{15}$$

Because the front is in motion, as before, the function v will <u>not</u> be discontinuous at the front, and the velocity c will be the trigger velocity associated with $v = v^+(t)$. From (15), we now have

$$\dot{v}^+ = -mc(v^+) + 1 \equiv H(v^+) \quad . \tag{16}$$

If there exists a value v^* such that $c(v^*) = \frac{1}{m}$, it will be a rest point for (16), and since c is an increasing function of v, $H'(v^+) < 0$, implying that v^* is a global attractor. Of course $v^* > v_0$, since $c(v^*) > 0$. Thus, the value $v = v_0$ is never achieved at the front, and the front does not divide in this case.

On the other hand if $\frac{1}{m}$ is so large that there is no trigger velocity equal to it, then $H(v) > 0$ for all $v \in (v_0,\bar{v})$. A little further analysis, which we shall not detail, then shows that v^+ grows till it reaches $\bar{v}$, at which point the front accelerates and becomes a "phase front" [5], v^+ remaining at $\bar{v}$.

Now suppose ψ' is not constant, but rather $\psi \to$ constant as $x \to \infty$, so $\psi' \to 0$. Again, it is clear that $\dot{v}^+ = \psi'c + 1 > 0$, and v^+ increases to $\bar{v}$, where it stays.

All this was under the assumption that $\psi' < 0$. On the other hand if $\psi' > 0$, (15) shows that no rest point exists, and v^{+} increases for a time. It never reaches $\bar{v}$, however, because v is increasing as a function of x, and before v^{+} can reach $\bar{v}$, there will be a value of $x > y(t)$ at which $v = \bar{v}$. At such a point, a downjump wave front (as shown in a similar setting in [4,5]) is generated. It moves to the left and eventually annihilates the original front. Again, no front division occurs.

Up to now, it has been assumed that the characteristic spatial scale of the initial data is that associated with the width of the front: so ϕ and ψ are well-behaved functions of $\frac{x}{\epsilon} = \xi$. This, for example, would come about if, in the original equation (1), $\alpha = 1$ and ϕ and ψ are functions of x. Now suppose the initial data have natural scale β when $t = 0$:

$$u = \phi(x/\beta), \quad v = \psi(x/\beta) \quad .$$

The possible effects when $\beta \neq \epsilon$ are seen mainly in Stage 1. If $\beta << \epsilon$, there is no effect at all; the initial function will vary more sharply than the front profile, and equilibration to the front's profile will still occur in a time interval $O(\epsilon)$. (This was true, in fact, in [12], where the initial data was discontinuous, corresponding to $\beta \to 0$.) On the other hand, the initial sharp variation in v may be such as to make a front division imminent, depending, of course, on the initial data. The arguments above show that for this to happen, the initial v distribution should be such as to make the characteristic front velocity zero or approximately so. This, again, is a special condition.

If $\beta >> \epsilon$, however, (assuming the phase image of the initial data crosses the ascending branch in Figure 1 at [illegible]), the equilibration in Stage 1 will be delayed. This is because on the spatial scale [illegible] associated with the front's equilibrium profile, u will be nearly constant and equal to u_0. Since u_0 is a rest state for u in (9)

(though an unstable one), it will take a long time for u to leave a small neighborhood of u_0, resulting in the mentioned delay. This delay will not enhance the process of the front's division, and may in fact prevent it.

Finally, some comments should be made about the special form of (2), namely that v does not diffuse and that g is a discontinuous function of u. Both of these provisions were for the purpose of making the analysis simpler and to provide for v's sharp variation. The sharp variation will still be the case if v is allowed to diffuse with a small diffusion coefficient, $\leq O(\varepsilon^2)$, or if g is allowed to be continuous, but with a large enough gradient near $u = u_0$. We shall not pursue the details. In [2] and [3], in fact, calculations were performed with a continuous g. If v diffuses by a relatively large amount, then it will not be able to achieve the variation $O(1)$ within the confines of a narrow front, and so will not support a division.

5. Discussion

The phenomena associated with Yakhno's mechanism are striking and suggestive, but rather special, as they depend on a great deal of symmetry in the basic equations. A disruption of this symmetry will generally destroy even the qualitative features of the phenomenon. In order to serve as a realistic model of leading centers in biological contexts, the basic model will have to be supplemented by other mechanisms, possibly through additional reaction-diffusion equations, which will serve to restore structural stability.

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A reaction-diffusion model suggested by Yakhno as a model for the centers of periodic activity often seen in excitable media such as cardiac tissue and special chemical reagents is subjected to multiple-scale analysis. A fairly complete description of the process (which involves the generation of wave fronts) is obtained, but it is shown to be structurally unstable as it depends on symmetries in the basic equations. The allowed amount of deviation from symmetry is estimated.

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